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Behaviour Approximated on Subgroups

J W Sanders

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G. M. Reed, Director

The United Nations University

UNU-IIST

International Institute for Software Technology

P.O. Box 3058
Macao

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Abstract

The recovery of behaviour from its approximation over substructures is fraught with pathology. Here the extent is considered to which the behaviour of a continuous function on a locally compact Abelian group can be approximated by its behaviour on proper closed subgroups. Known results are summarised when the behaviour concerns integrability and the group is the circle; then boundedness and other limiting behaviour 'at infinity' are considered for more general groups. It is shown that if a continuous function is bounded on each proper closed subgroup of a connected locally compact Abelian group then it is bounded on the whole group. As befits this *Festschrift*, the techniques are predominantly topological. In passing we reflect on criteria for the difficult problem of identifying 'substructures' in Computer Science.

Dedicated to Peter Collins and Mike Reed
on the occasion of their joint Festschrift
in Topology and Computer Science

Jeff Sanders is Principal Research Fellow at UNU-IIST, having recently joined from the Programming Research Group at Oxford. His interests lie largely in Formal Methods.

Contents

1 Introduction

Mathematics takes for granted modularisation of structure, but Computer Science has been slow to exploit it. And how else can the behaviour of complicated systems on the scale experienced in Computer Science be understood, and such systems constructed correctly? Perhaps part of the difficulty in Computer Science is that much of the interesting behaviour is dynamic (either of a single process in execution—for a typical example see Mike Reed's [16]—or of a single system evolving from specification to implementation [6]); so the appropriate substructures have taken time to crystallise. Since Computer-Science structures are discrete, and substructures are obtained by hiding of information, the appropriate notions of substructure may well not look like those with which we have been trained in traditional Mathematics.

For example, early attempts to define the timed behaviour of communicating concurrent processes did so with a hierarchy of models connected by injections. However nowadays such a hierarchy would be required to be based on Galois connections rather than the more traditional injections, reflecting the importance of the refinement partial orders in each structure in the hierarchy, and the need for preserving laws along the hierarchy. Though old, Galois connections have not in the past been included in the syllabus of the average mathematics student; injections have. In that example the information being abstracted along the hierarchy concerns process behaviour. The novelty arises because of nondeterminism (on which the refinement order is based), which alone has accounted for much development in (computing-related) Mathematics.

Another example illustrating the importance of information hiding is provided by the spatial logics, and separation logic in particular, that have been found useful in reasoning about pointer structures at a level of abstraction able to hide unnecessary implementation detail but still to capture mutability. They have no counterpart in traditional logic. We return to this issue of substructure in Section 7, but for now resolve to pay more attention than is usual to justifying our choice of substructure in this paper.

One structure important for both Mathematics and Computer Science has been the space consisting of the countable power of a doubleton set under the product topology, with either the discrete or Sierpinski topology on the doubleton. The Sierpinski topology has been used to model information increase [18], whilst the discrete topology has been useful in describing fundamental properties of reactive computations [1].

This paper adopts the mathematical view of structure, recognising as a group the discrete doubleton $\{\pm 1\}$ and hence also its power $\{\pm 1\}^{\mathbb{N}}$, the (totally disconnected, compact Abelian) Cantor group (for example, [4] Chapter 14). It considers the question:

> to what extent can the behaviour of a function over such a group be inferred from that of its restrictions to appropriate substructures?

In Section 2 we consider the circle and Cantor groups, and in Section 3 consider $\mathbb{R}^n$ and its extensions. But the first step consists of identifying those substructures.

Here we choose *closed subgroups*, for the following reasons. Firstly we wish to investigate the interaction between topology and algebra. Thus we consider each substructure both topologically and algebraically: as a topological group (and a subgroup trivially forms a topological group with the relative topology). If a subgroup is closed then its quotient is (trivially) Hausdorff; if a subgroup is locally compact then it is automatically closed; and anyway the closure of a subgroup is again a subgroup. Finally in the important cases of the Cantor group and the circle group, proper closed subgroups are finite and so their Haar measure is finite counting measure, whilst on the real line the Haar measure of the proper closed subgroups is discrete counting measure.

The answer to our question above for straight 'values' of the function on the Cantor group is uninteresting: the restriction of f to (proper) closed subgroups determines f iff f is continuous, since the union of those subgroups is dense in the group. Instead we consider 'derived' information about f, and over various groups.

The paper begins by considering integration and, broadening to the circle group, discusses a known result; the techniques appear to be algebraic in nature but turn out to hinge on analysis. It then broadens still further to locally compact Abelian groups and considers boundedness in Section 3 and other behaviour at infinity in Section 5; this time the results are new and the techniques predominantly topological.

2 Integration to zero

Suppose that f is a complex-valued function on the circle group, so that f can be thought of as a 1-periodic function on the unit interval $[0, 1]$ with the standard topology. What can be said about f if it integrates to 0 over each closed subgroup of the circle?

Recall (for example [10], Chapter 2, Corollary 2), that every proper closed subgroup of the circle is finite and hence has the form $S_n \;\widehat{=}\; \{j/n \mid 0 \leq j < n\}$ for some positive integer n. Since $\bigcup S_n$ is countable, and hence null, we assume f to be continuous (rather than lying merely in some L^p space where behaviour on a null set is incidental). The assumption that f integrates to 0 over each closed subgroup is thus

(1) $\quad \forall\, n \geq 1 \cdot \sum_{S_n} f \;=\; 0\,.$

Evidently functions f which are odd about 1/2 (*i.e.* $\forall\, x : [0, 1] \cdot f(x) + f(1-x) = 0$) have that property. So is f odd?

In general 'no', as demonstrated by an old example of Hille and Szasz [5]. The function

$$x \mapsto \sum_{m\geq 1} \frac{\mu(m)}{m}\, e^{2\pi i m x}$$

(where μ denotes the Möbius function; see for instance [2], Chapter 2) is continuous (by an estimate of Davenport [3]) but not odd (it is simple to see that a function is odd iff its Fourier transform is odd on $\mathbb{Z}$; and the Fourier transform of that function evidently vanishes for all negative, but not all positive, integers). But equation (1) is known to hold, as can be verified by a calculation using well-known properties (mostly facile but one deep) of μ:

$\sum_{0\leq j<n} \sum_{m\geq 1} \frac{\mu(m)}{m}\, e^{2\pi i m j/n}$

$=$ uniform convergence of the series [3]

$\sum_{m\geq 1} \frac{\mu(m)}{m} \sum_{0\leq j<n}\, e^{2\pi i m j/n}$

$=$ summing the geometric progression

$\sum_{m\geq 1} \frac{\mu(m)}{m}\, (n \textbf{ if } n\,|\,m \textbf{ else } 0)$

$=$ change of variable $m = nd$

$\sum_{d\geq 1} \frac{\mu(nd)}{nd}\, n$

$=$ μ multiplicative

$\mu(n) \sum_{d\geq 1} \frac{\mu(d)}{d}$

$=$ the prime number Theorem implies $\sum_{d\geq 1} \frac{\mu(d)}{d} = 0$; see for instance [2], p. 97

$0\,.$

However the following result is known [7, 8]:

Theorem 0. If a continuous function $f : [0,1] \to \mathbb{C}$ on the circle group sums to 0 over each proper closed subgroup then f is odd provided it has absolutely summable Fourier transform.

That result is indeed of the type that interests us here: discrete behaviour of f restricted to the various S_n is extended to the whole group (see Corollary 0). A proof is of some interest for the way in which it facilitates the passage from the S_n to the circle group. We outline here the proof from [7] that uses an infinite version of Möbius inversion based on what Mathematicians call a formal identity and Computer Scientists call an informal one, and whose justification requires something equivalent to a dominated-convergence argument. For details of the argument we refer to *loc. cit*, p. 261.

Using the characterisation above (f odd iff $\hat{f}$ odd) to switch to the Fourier transform, the identity in question is

$$\forall\, i : \mathbb{Z} \cdot \hat{f}(-i) + \hat{f}(i) \;=\; \sum_{i\mathbb{Z}} \hat{f} - \sum_{2i\mathbb{Z}} \hat{f} - \sum_{3i\mathbb{Z}} \hat{f} + \sum_{6i\mathbb{Z}} \hat{f} \;\; \cdots \;.$$

The right-hand side is proved to be 0 by observing that its terms evaluate the sum on the left according to a certain summability method, defined by the sequence of functions

$$\psi_p \;\widehat{=}\; \sum_{n \in \Pi(p)} \mu(n)\xi_n \,,$$

where ξ_n denotes the characteristic function of $ni\mathbb{Z}$ and where, for prime p, $\Pi(p)$ denotes the set of products of distinct primes each at most p:

$$\Pi(p) \;\widehat{=}\; \{p_1 p_2 \ldots p_k \mid \exists\, k : \mathbb{N} \cdot p_1, p_2, \ldots, p_k \text{ are distinct primes } \leq p\}.$$

For then standard properties of the Möbius function (as given in say [2], Theorem 2.1) serve to show that $\hat{f}\psi_p$ tends pointwise to $\hat{f}$ times the characteristic function of the doubleton $\{-i, i\}$. So by the Dominated-Convergence Theorem (for which a good reference, appropriate to this *Festschrift*, is [17], Theorem 1.34) for the discrete space $l^1(\mathbb{Z})$, as p increases

$$(2) \quad \sum_{\mathbb{Z}} \hat{f}\psi_p \;\rightarrow\; \hat{f}(-i) + \hat{f}(i).$$

But, again by [2] Theorem 2.1, each term on the left of (2) is 0 and hence so too is the right-hand side, as required. □

As a consequence we see that behaviour over the whole group is determined by that over proper closed subgroups; the proof follows immediately by definition of 'odd function'.

Corollary 0. If a function on the circle group has absolutely summable Fourier series (and so is continuous) and sums to 0 over each proper closed subgroup then it integrates to 0 over the whole group.

Reflecting on the proof of Theorem 0, some non-trivial step is required to handle the limit (there, summability via the ψ_p and use of the Dominated-Convergence Theorem; in the alternative proof of [8], summability and convergence by a Tauberian theorem of Hardy). What happens in the case of the (usually simpler) Cantor group, a situation nearer to the hearts of Computer Scientists? Each element x (other than the identity) has order 2, so the set $\{e, x, x^{-1}\}$ consisting of the identity, the element x and its inverse (and so having cardinality either 1 or 3) is a closed subgroup (in the product topology). So if a function f integrates to 0 over each closed subgroup, then for all x it satisfies $f(x^{-1}) + f(x) = 0$, and so f is odd. The converse is trivial.

3 Boundedness

We turn to investigate the case of f defined over a Cartesian space $\mathbb{R}^n$ with substructures the proper closed subgroups. When is behaviour of f over those substructures sufficient to imply it over the whole of $\mathbb{R}^n$? In moving from $[0,1]$ to the locally compact $\mathbb{R}^n$, the behaviour we consider is functional behaviour 'at infinity'.

We begin by identifying (to within topological isomorphism) the closed subgroups of Cartesian space $\mathbb{R}^n$, following the method of say [10], Chapter 2. Any closed subgroup of $\mathbb{R}^n$ consists of a nondiscrete part and a discrete part, configured in $\mathbb{R}^n$ according to some basis $\{r_i \mid 0 \leq i < n\}$ as follows. The nondiscrete part consists of the span (the set of linear combinations with real coefficients) of part of the basis: of $\mathit{span}\{r_i \mid 0 \leq i < a\}$; the discrete part consists of the additive subgroup generated by the remaining part of the basis: of $\mathit{group}\{r_i \mid a \leq i < b\}$, where $0 \leq a \leq b \leq n$. The closed subgroup is then (isomorphic to) the group generated by the union of those parts

$$\begin{aligned} & \mathit{group}(\mathit{span}\{r_i \mid 0 \leq i < a\} \cup \mathit{group}\{r_i \mid a \leq i < b\}) \\ = \; & \mathit{group}(\mathit{span}\{r_i \mid 0 \leq i < a\} \cup \{r_i \mid a \leq i < b\}). \end{aligned}$$

Next we observe, by the Principal Structure Theorem for locally compact Abelian (LCA) groups, [10], Theorem 26, that a connected LCA group is isomorphic, as a topological group, to $\mathbb{R}^n \times C$ with the product topology, where n is a natural number and C is a connected compact Abelian group. Thus a closed subgroup of an arbitrary connected LCA group is isomorphic to

$$(3) \quad \mathit{group}(\mathit{span}\{r_i \mid 0 \leq i < a\} \cup \{r_i \mid a \leq i < b\}) \times C$$

where $\{r_i \mid 0 \leq i < n\}$ is a basis of $\mathbb{R}^n$, $0 \leq a \leq b \leq n$ and C is a compact Abelian group.

In fact we need be concerned with only one kind of subgroup. We define a *d-subgroup* of $\mathbb{R}^n \times C$, where C is a compact Abelian group, to be a closed subgroup in which the basis is the standard basis, $\{e_i \mid 0 \leq i < n\}$, the nondiscrete part is empty, $a = 0$, and the discrete part uses all of the basis: $b = n$. Writing $r : \mathbb{R}^n$ for the tuple of generators, the d-subgroup $H(r)$ is

$$H(r) \;\; \widehat{=} \;\; \mathit{group}\{r_i e_i \mid 0 \leq i < n\} \times C.$$

For example any d-subgroup of $\mathbb{R}$ consists (to within isomorphism) of the multiples of some nonzero real: it has the form $r\mathbb{Z}$ for $r \neq 0$. For $\mathbb{R}^2$ it has the form $r\mathbb{Z} \times s\mathbb{Z}$ for nonzero reals r and s.

The set of d-subgroups of a given connected LCA group is closed under neither union nor intersection. That latter fact will require us to do some simultaneous approximation of d-subgroups in the theorems to follow.

The main result of this section is:

Theorem 1. Suppose that G is a connected LCA group and f is a continuous complex-valued function on G whose restriction to each d-subgroup of G is bounded. Then f is bounded on G.

Our proof reduces to converting boundedness over d-subgroups of $\mathbb{R}^n$ to boundedness over $\mathbb{R}^n$, and the main step permitting that passage from the discrete to the nondiscrete is the following lemma. In it we write $B_\varepsilon(x)$ for the open ε-ball centred at $x : \mathbb{R}^n$.

$$B_\varepsilon(x) \ \widehat{=}\ \{y : \mathbb{R}^n \mid \|x - y\| < \varepsilon\}$$

Lemma 1. For any $b : \mathbb{R}^n$ and any $\varepsilon > 0$, the set

$$(4) \quad \bigcup \{H(r) \mid \ r \in B_\varepsilon(b)\}$$

is a neighbourhood of infinity in $\mathbb{R}^n$.

Proof. We have to show that for some $R > 0$, the set (4) covers $\mathbb{R}^n \setminus B_R(0)$. For $0 \leq i < n$ and $m : \mathbb{Z}$, the set

$$\{mr_i \mid \ r = (r_j)_{0 \leq j < n} \in B_\varepsilon(b)\}$$

is an interval in $\mathbb{R}$ which is centred at mb_i and has length $2m\,|\,b_i - \varepsilon\,|$. Consequently there is a positive integer M_i so that whenever $m > M_i$, those intervals overlap. Hence their union covers $\mathbb{R} \setminus [-M_i b_i, M_i b_i]$.

It remains to set $R \ \widehat{=}\ (\sum_{0 \leq i < n} (M_i b_i)^2)^{\frac{1}{2}}$ in order to conclude that the set (4) covers $\mathbb{R}^n \setminus B_R(0)$ as desired. □

Proof of Theorem 1. We have already observed that any connected LCA group is isomorphic to $G \ \widehat{=}\ \mathbb{R}^n \times C$ with the product topology where n is a natural number and C is a compact Abelian group. Suppose that f is a continuous complex-valued function on G.

The hypothesis means that the following function is well defined

$$\phi : \mathbb{R}^n \to \mathbb{R}$$
$$\phi(r) \ \widehat{=}\ \|f \restriction H(r)\|_\infty .$$

Firstly we show that ϕ is lower semicontinuous:

$$\forall\, t : \mathbb{R} \cdot \{r : \mathbb{R}^n \mid t < \phi(r)\} \text{ is open in } \mathbb{R}^n.$$

Accordingly, suppose that $r : \mathbb{R}^n$ and $t : \mathbb{R}$ satisfy $t < \|f \restriction H(r)\|_\infty$. By definition of supremum there is some element $s = (n_i r_i)_{0 \le i < n}$ of $H(r)$ with $\mid f(s) \mid$ close to $\|f \restriction H(r)\|_\infty$. Then we reason

$$
\begin{array}{ll}
t & \\
< & \text{on some neighbourhood } B_\varepsilon(s) \text{ of } s\text{, by continuity of } f \\
\|f \restriction B_\varepsilon(s)\|_\infty & \\
\le & A \subseteq B \;\Rightarrow\; \|f \restriction A\|_\infty \le \|f \restriction B\|_\infty \\
\|f \restriction \bigcup\{H(v) \mid v \in B_\varepsilon(s)\}\|_\infty & \\
= & s \mathrel{\widehat{=}} (n_i r_i)_{0 \le i < n} \\
\|f \restriction \bigcup\{H(v) \mid v \in B_\varepsilon((n_i r_i)_{0 \le i < n})\}\|_\infty & \\
\le & H((n_i w_i)_{0 \le i < n}) \subseteq H((w_i)_{0 \le i < n}) \text{ for any integers } n_i \ne 0 \\
\|f \restriction \bigcup\{H(v) \mid v \in B_\varepsilon((r_i)_{0 \le i < n})\}\|_\infty & \\
= & \text{definition of } \phi \\
\phi \restriction B_\varepsilon((r_i)_{0 \le i < n}) &
\end{array}
$$

as required. (Note that without supremum in the definition of ϕ the present argument, and the lower semicontinuity it establishes, fails).

So, for $m : \mathbb{N}$, letting

$$F_m \mathrel{\widehat{=}} \{r : \mathbb{R}^n \mid \phi(r) \le m\},$$

each F_m is closed with $\bigcup_{0 \le m} F_m = \mathbb{R}^n$. So, by the Baire category theorem ([17], Theorem 5.6), some F_M contains an open ball $B_\varepsilon(b)$. By Lemma 1 there exists $R > 0$ such that

$$\bigcup\{H(r) \mid r \in B_\varepsilon(b)\}$$

covers $\mathbb{R}^n \setminus B_R(0)$. Hence,

$$\|f \restriction (\mathbb{R}^n \setminus B_R(0)) \times C\|_\infty \le M.$$

But f is continuous on the closure of $B_R(0) \times C$, and so is bounded there too. So f is bounded on G. $\square$

Corollary 1. If a complex-valued continuous function is bounded away from 0 on each d-subgroup of a connected LCA group G then it is bounded away from 0 on G.

Proof. If the function is f, it suffices to take the pointwise inverse $1/f$ in Theorem 1. □

4 Optimality

Theorem 1 cannot be extended to cover disconnected groups, nor discontinuous functions, as shown by the following two examples.

Example 1. Consider the additive group $\mathbb{Z}$ of integers with the discrete topology and define a (continuous) function $f : \mathbb{Z} \to \mathbb{C}$ by its restrictions

$$\begin{array}{rcl} f \restriction 2\mathbb{Z} & \widehat{=} & 2 \\ f \restriction (3\mathbb{Z} \setminus 2\mathbb{Z}) & \widehat{=} & 3 \\ & \vdots & \\ f \restriction (p_n\mathbb{Z} \setminus \cup_{1 \le j \le n}\ p_j\mathbb{Z}) & \widehat{=} & n, \end{array}$$

where p_j denotes the jth prime, and to be 0 at ± 1. Evidently any d-subgroup of $\mathbb{Z}$ has the form

$$(p_1^{m_1}.p_2^{m_2}.\ \ldots\ .p_s^{m_s})\,\mathbb{Z}, \qquad \text{with some } m_i \text{ non-zero,}$$

and on that subgroup f is at most i. Hence f is bounded over each d-subgroup of $\mathbb{Z}$ but is unbounded on $\mathbb{Z}$ itself. □

Example 2. Consider the additive group $\mathbb{R}$ and let $\{r_n \mid n \in \mathbb{N}\}$ be an infinite set of rationally independent real numbers. Define f by its restrictions

$$f \restriction r_n\mathbb{Z} \ \widehat{=} \ n$$

and to be 0 elsewhere on $\mathbb{R}$. Now each d-subgroup of $\mathbb{R}$ has the form $r\mathbb{Z}$ for some $r \in \mathbb{R}$ and hence f is bounded on each of those. However f is unbounded on $\mathbb{R}$. □

5 Convergence at infinity

In this section we refine Theorem 1 to reflect more refined behaviour of f 'at infinity'.

For a LCA group G and $c : \mathbb{C}$, let $C(G, c)$ denote the set of continuous complex-valued functions on G that tend to c at infinity: for each $\varepsilon > 0$, $\{x : G \mid \ |f(x) - c| > \varepsilon\}$ is compact.

The characteristic function of the rationals shows that a function on $\mathbb{R}$ can be constant on each d-subgroup without being constant on the whole group. But if the function is continuous, we have the following theorem. (In fact it is not required for the function to tend to a limit on *each* d-subgroup; in $\mathbb{R}$ we require just the d-subgroups generated by nonzero rationals.)

Theorem 2. Suppose G is a connected LCA group, f is continuous on G and for each d-subgroup H of G there exists $c_H : \mathbb{C}$ such that the restriction $f \restriction H$ belongs to $C(H, c_H)$. Then for some $c : \mathbb{C}$, $f \in C(G, c)$.

Proof. The first step is to show that the limit values over d-subgroups all coincide. Consider first the case $G \ \widehat{=}\ \mathbb{R}$, where d-subgroups take the form $r\mathbb{Z}$ with $r : \mathbb{R} \setminus \{0\}$. The function

$$\begin{array}{l} \phi : \mathbb{R} \to \mathbb{R} \\ \phi(r) \ \widehat{=}\ \lim_{|n| \to \infty} f(rn) \end{array}$$

is constant on the (nonzero) rationals. For suppose $r = a/b$ and $s = c/d$. Then integral multiples of ac occur in both $r\mathbb{Z}$ and $s\mathbb{Z}$ hence the limit of f, which exists on the d-subgroup $ac\mathbb{Z}$, must equal the limits of f on $r\mathbb{Z}$ and $s\mathbb{Z}$. In other words, $\phi(r) = \phi(s)$.

Now suppose that $t : \mathbb{R}$ is irrational. We show $\phi(t) = \phi(1)$ and so deduce that ϕ is constant. Suppose $\varepsilon > 0$. By rational approximation to t (traditionally established by the continued fraction expansion of t in which $|\ t - b/a\ | < 1/a^2$), there are integers a, b for which $|\ at - b1\ | < \varepsilon$. (Expressed informally, $t\mathbb{Z}$ gets arbitrarily close to $1\mathbb{Z}$, infinitely often. More is true, in view of the uniform distribution of $t\mathbb{Z}$ reduced modulo 1, but we do not need it.) So, by continuity of f, $\phi(t) = \phi(1)$.

Next consider $G \ \widehat{=}\ \mathbb{R}^n$. The previous arguments apply coordinatewise, at each $i : [0, n)$. Firstly they show that given two d-subgroups $H(r)$ and $H(s)$ with r_i and s_i rational, the d-subgroup $H((r_i s_i)_{0 \le i < n})$ establishes $\phi(r) = \phi(s)$. Secondly given a d-subgroup $H(t)$ with some t_i irrational, approximation along axis i and continuity of f (together with the previous technique or result for axes i with rational t_i) shows that f tends to the same limit at $H(t)$.

So we may assume that the common limit value over all d-subgroups is c. We now show by contradiction that $f \in C(\mathbb{R}^n, c)$. If not, then there is some $\varepsilon > 0$ and some unbounded sequence $(x_k)_k$ of points in $\mathbb{R}^n$ for which for all k, $|\ f(x_k) - c\ | > \varepsilon$. By continuity of f that inequality continues to hold on some open ball about x_k; let T denote the union (as k varies) of those open balls. We gain a contradiction by finding $r \in \mathbb{R}^n$ for which the d-subgroup $H(r)$ intersects T infinitely often, showing that $f \notin C(H(r), c)$. We do so with the following lemma, proved for $\mathbb{R}$ by Priestley [14]. In it, $No(r)$ denotes the number of times, possibly infinite, that the d-subgroup $H(r)$ intersects T.

Lemma 2. Suppose $r : \mathbb{R}^n$ and the d-subgroup $H(r)$ intersects T a finite number of times. Then on some neighbourhood of r the function No is bounded below by $No(r)$ yet takes arbitrarily large values.

Proof. Let $J_1, J_2, \ldots, J_{No(r)}$ be the balls in T that intersect $H(r)$. Since they are open, there is a neighbourhood J of r such that if $s \in J$ then $H(s)$ also intersects $J_1, J_2, \ldots, J_{N(r)}$. That is, $No(s) \geq No(r)$.

Now applying Lemma 1, the set $\cup_m mJ$ intersects a new ball, J_ν, in T; hence we can choose $t \in J$ for which the d-subgroup $H(t)$ intersects $J_1, J_2, \ldots, J_{No(r)}, J_\nu$. Thus $No(t) \geq No(r) + 1$. Replacing r by t in the proof above yields a neighbourhood K of t contained in J, and a point $u \in K$ for which $No(u) \geq No(t) + 1$. Proceeding inductively we see that N takes arbitrarily large values on J. □

Returning to the proof of Theorem 2, Lemma 2 shows that for positive integers m, the sets

$$F_m \ \widehat{=}\ \{r : \mathbb{R}^n \mid \ No(r) \leq m\}$$

are closed and nowhere dense in $\mathbb{R}^n$. Hence their union is a set of first category in $\mathbb{R}^n$ and so its complement is of second category. In particular there is an $r : \mathbb{R}^n$ with $No(r) = \infty$. That is, the d-subgroup $H(r)$ intersects T infinitely often.

Finally, the proof above can be reworked to incorporate the product with a compact Abelian and connected group C since, being compact, it does not affect the behaviour of f 'at infinity'. The proof of Theorem 2 is complete. □

6 Notes

The examples of Section 4 show that Theorem 2 cannot be extended to disconnected groups nor to discontinuous functions.

For the special case $G \ \widehat{=}\ \mathbb{R}$, Theorem 2 extends folk-lore (see [14]); indeed our proof has been an extension of the proof for $\mathbb{R}$. In that setting we infer the following corollary, in which the first case is known ([15], Problem 10, p. 38). The proof follows from the statement or proof of Theorem 2.

Corollary 2. Suppose that f is a continuous complex-valued function on $\mathbb{R}$.

1. If $f \restriction r\mathbb{Z} \in c_0(r\mathbb{Z})$ for all r in a set of second category, then $f \in c_0(\mathbb{R})$.

2. If f is bounded away from 0 on each $r\mathbb{Z}$, then it is bounded away from 0 on $\mathbb{R}$.

3. If $f \restriction r\mathbb{Z}$ is decreasing for each rational r, then f is decreasing.

7 Conclusion

This paper has considered the problem of promoting the behaviour of a continuous function from proper closed subgroups of a LCA group to the group itself. After setting the scene by considering known results concerning the circle group and behaviour consisting of 'integrability to 0', it has considered connected LCA groups and behaviour consisting of 'boundedness' and then 'limiting behaviour at infinity'. In each case a Baire category argument has sufficed as the basic tool to infer that the same behaviour persists over the whole group; furthermore, only certain (d-subgroup) substructure need be assumed.

The choice of closed subgroup seems entirely appropriate in the first part of the paper, both theoretically and in supporting applications (for example approximate integration [7, 8] where equally-spaced points of quadrature form a closed subgroup). But in the second part it does not, as indicated by the appearance of d-subgroups. There are many subregions of the Euclidean spaces $\mathbb{R}^n$ which could have been used instead; that is left for further work.

Returning to the question of substructures in Computer Science, the answer must depend—as in Mathematics—on the kind of system and question under consideration. But to date little progress has been made. Common to most situations is the approximation of a discrete system by those in which information has been hidden (in exactly the same way that in Mathematics $\exists x \cdot P$ approximates P). At the same time the 'approximation' relation (or, in reverse, the refines/implements relation) forms a partial order on the space of structures, and the choice of substructure should ensure that the behaviours over a refinement relate to those over the whole space by a Galois connection. Moves in that direction are being made (for example [9, 11]) but we still have some way to go in spite of early warnings [12, 13].

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